

Quantum Hamiltonian complexity and the detectability lemma

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Abstract

Local Hamiltonians, the central object of study in condensed matter physics, are the quantum analogue of CSPs, and ground states of Hamiltonians are the quantum analogue of satisfying assignments. The major difference between the two is the existence of multi-particle entanglement in the ground state, which introduces a whole new level of difficulty in tackling questions such as quantum PCP, quantum analogues of amplification, etc.

The Lieb-Robinson bound is a sophisticated analytic tool used in condensed matter physics for handling quantum correlations in ground states, by bounding the velocity at which disturbances propagate in quantum local systems. In this paper we show that the detectability lemma (introduced in a different context in Ref. [1]), when viewed from the right perspective, can be used in place of the Lieb-Robinson bound for the rich case of frustration free Hamiltonians. The advantage of this is that the resulting proofs are simpler and more combinatorial, and may be generalizable to solve some of the most fundamental questions in Hamiltonian complexity. Additionally, we give an alternative proof of the detectability lemma, which is not only simple and intuitive, but also removes a key restriction in the original statement, making it more suitable for this new context.

Specifically, we use the detectability lemma to give a simpler proof of Hastings' seminal 1D area law [2] for frustration-free systems. Proving the area law for two and higher dimensions is one of the most important open questions in Hamiltonian complexity, and the combinatorial nature of the detectability lemma based proof and the resulting simplification holds out hope for a possible generalization. We also provide a one page proof of Hastings' proof that the correlations in the ground states of gapped Hamiltonians decay exponentially with the distance (once again, restricted to frustration-free systems). We argue that the detectability lemma in this form constitutes a basic tool for the study of local Hamiltonians and their ground states from a computational point of view.

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1 Introduction

Local Hamiltonians and ground states, the central object of study of condensed matter physics, are the quantum analogues of the central objects of study in computational complexity: constraint satisfaction problems (CSP) and their satisfying assignments. This connection, which ties together two seemingly very different areas, is the starting point for the emergence of the new field, Quantum Hamiltonian Complexity, in which properties of local Hamiltonians and ground states are being studied from a computational complexity point of view. Over the past few years, this direction has shed exciting new insights into quantum information theory as well as into quantum physics. Of crucial importance here is the difference between the quantum and classical domains: the quantum analogue of the satisfying assignment, namely the ground state, can exhibit extremely intricate multi-particle entanglement. This additional player in the game makes borrowing results from the classical domain to the quantum domain extremely challenging, cf. the wide open major open problem of whether a quantum analogue of PCP holds [1]; it also opens up completely new directions of research regarding the entanglement properties of ground states of local Hamiltonians.

General quantum states require 2^n complex numbers to describe. One of the major goals of quantum Hamiltonian complexity is to derive bounds on the entanglement exhibited in ground states of interesting classes of local Hamiltonians; the purpose of those bounds and restrictions on the entanglement is to lead to an efficient description and analysis of ground states in cases of interest. There is a beautiful sequence of papers using structures called tensor networks, with special cases such as MPS [3, 4, 5, 6], PEPS [7], TN [8], and MERA [9], which provide such efficient descriptions in certain cases.

Area laws constitute one of the most important tools for bounding entanglement in such systems. Consider the interaction graph (hypergraph) associated with a local Hamiltonian – it has a vertex for each particle and an edge for each term of the Hamiltonian. Intuitively and very roughly, an area law says that entanglement is local in this interaction graph in the following sense: consider a subset of particles L . Then the entanglement between L and \bar{L} in the ground state is locally “concentrated” along the edges between L and \bar{L} ; more precisely, the area law states that the entanglement entropy across the cut is big-Oh of the number of edges crossing between L and \bar{L} . This is clearly a very strong restriction on the entropy, which in the general case would be of order of the number of particles (nodes) in L . Proving area laws for typical classes of Hamiltonians is thus a holy grail in quantum Hamiltonian complexity.

A few years ago, in a seminal paper [2], Hastings proved that the area law holds for 1D systems (i.e., when the interaction graph is a path), for gapped Hamiltonian – that is, Hamiltonians whose overall spectral gap is of order $\mathcal{O}(1)$. In this case, the area law says that ground state entanglement across any contiguous cut is bounded by a constant. From this, one can deduce that the ground state of such systems can be described efficiently (by an MPS of polynomial bond dimension – see Ref. [2]). The question of whether ground states in two and higher dimensions obey an area law is still wide open.

Hastings’ proof of the 1D area law, and many other proofs related to entanglement and correlations in ground states, use sophisticated analytic methods. Perhaps the most important of those is the famous

Lieb-Robinson bound (LR bound) [10, 11], which bounds the velocity at which disturbances propagate in quantum local systems; Fourier analysis, and other techniques are important players too. These analytic tools constitute a major barrier for a fuller participation by computer scientists in this important aspect of Hamiltonian complexity. Also, these analytic techniques seem to inherently involve the dynamics of the system in time, according to the Hamiltonian. However, purely from an aesthetics point of view, it should be possible to explain kinematic results about the ground state without resorting to dynamical arguments (which is what the LR bound is). Or, in other words, without adding the extra dimension of time to the problem. In addition, the kinematic problem seems, at least on the surface, to be of a combinatorial nature, thereby suggesting a combinatorial solution.

In this paper we introduce a combinatorial tool to tackle the above mentioned problems, and, in particular, to get a handle on correlations and entanglement in ground states of local Hamiltonians. This is a simple, basic version of the detectability lemma of Ref. [1]. We demonstrate that when the system is frustration-free, many of the results that rely on the traditional analytic tools can be obtained in a much simpler, direct and intuitive way using this tool; we argue that the detectability lemma in this form constitutes a basic tool for the study of local Hamiltonians and their ground states from a computational point of view.

Our starting point is the Detectability Lemma (DL) introduced in Ref. [1]. There, the motivation for the DL was quite specific: to help translate classical results about CSPs to quantum results about local Hamiltonians. It was used to prove a quantum analog of gap amplification (a component of Dinur’s proof of the PCP theorem [12]). The DL made it possible to sensibly make a statement of the form “If the ground state energy is at least k then the probability that it violates at least ck terms of the Hamiltonian is bounded below by a constant”. The DL of Ref. [1] holds under the mild assumption (which is essentially true in most interesting cases) that each particle participates in a bounded number of terms of the Hamiltonian, and therefore the terms of the Hamiltonian can be partitioned into a constant number of layers, each consisting of terms acting on disjoint sets of particles. Ref. [1] also required an additional technical assumption, that the number of distinct types of terms of the Hamiltonian are bounded.

Here, we reformulate the DL and put it in a much broader and basic context. Our reformulation of the DL asks the following question: consider a gapped frustration-free local Hamiltonian $H = \sum_{i=1}^m H_i$ with $0 \leq H_i \leq 1$. i.e., the ground energy of H is 0, and the spectral gap is $\epsilon = \mathcal{O}(1)$. The frustration-free assumption means that the ground state minimizes the energy of *every* local term, so no term is “frustrated”. Can we approximate the projection Π_{gs} on the ground state, $|\Omega\rangle$, by a “local” operator? Such a local approximation would be extremely useful, as it would enable deducing local properties of the ground states such as area laws and decay of correlations. Indeed, such an approximation of a projection on the ground state is essentially what is done by the traditional analytic tools that use the LR bound, as we explain in Sec. 4. The approximation offered by the DL, however, has more of a combinatorial flavor, and is therefore much easier to handle.

A natural first guess of such a local approximation of Π_{gs} is the positive semi-definite operator $G \stackrel{\text{def}}{=} (\mathbb{1} - \frac{1}{m}H)$, where m is the number of terms in the Hamiltonian. G fixes the ground state, and

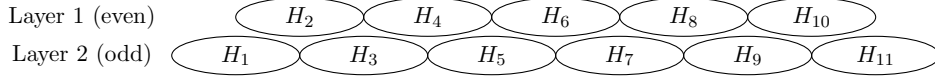


Figure 1: An illustration of a 1D system of two-local, nearest neighbors, interactions. The local terms (H_1, H_2, \dots) can be arranged in two layers (even and odd), such that the terms in each layer do not overlap.

shrinks all the orthogonal space to it by a factor; however the shrinkage is very limited, by a factor of $(1 - \epsilon/m)$. To get a good approximation, one would need to apply this operator polynomially many times, and by this we would lose the locality of the operator. Indeed, the expression $(\mathbb{1} - H/m)^m$ contains products of m overlapping terms whose overall support is of the order the size of the system. Our challenge is therefore to get a local operator that preserves the ground state but shrinks the orthogonal subspace by a constant factor, rather than by ϵ/m . For simplicity of presentation in the introduction, let us consider the simplest scenario, in which the particles are set on a 1D chain, and the interactions are two local. Denote by P_i the projection on the ground state of the terms H_i . Notice that the terms in the Hamiltonian can be partitioned into two layers, the even and odd terms, each acting on disjoint sets of particles (see Fig. 1);

Denote by Π_{odd} the product of the projections on the ground spaces of all odd terms P_1, P_3, \dots and by Π_{even} the product for the even terms. Then the operator $A \stackrel{\text{def}}{=} \Pi_{\text{odd}} \Pi_{\text{even}}$ is the “local” operator we want. The DL states:

Lemma 1.1 (Detectability Lemma (DL) in 1D) *Let $A \stackrel{\text{def}}{=} \Pi_{\text{odd}} \Pi_{\text{even}}$, and let \mathcal{H}' be the orthogonal complement of the ground space. Then*

$$\|A|_{\mathcal{H}'}\| \leq \frac{1}{(\epsilon/2 + 1)^{1/3}}. \quad (1)$$

The DL says that the application of A to any vector moves the vector closer to the ground state of H by cutting down the mass in the orthogonal subspace by a constant factor. This implies that Π_{gs} , the projection into the ground space of H , can be approximated to within exponentially good precision by applying the operator A ℓ times: $\Pi_{\text{gs}} = A^\ell + e^{-\mathcal{O}(\ell)}$.

Let us explain why this operator is indeed “local”. When A is applied ℓ times to some local perturbation B that acts on the ground state $|\Omega\rangle$, there is a pyramid-shaped “causality cone” of projections that is defined by B . These are simply all terms which are graph-connected to the operator B (see Fig. 2). All the projections outside that cone commute with B and can therefore be absorbed in the ground state (since $P_i|\Omega\rangle = |\Omega\rangle$), leaving us with a *local* operator of support of size $\mathcal{O}(\ell)$. Effectively, A^ℓ acts non-trivially only on a region of width $\mathcal{O}(\ell)$, when applied to $B|\Omega\rangle$.

We give here a new simple proof of this reformulation of the DL, in the process dropping the assumption of Ref. [1] about the number of distinct types of terms of the Hamiltonian.

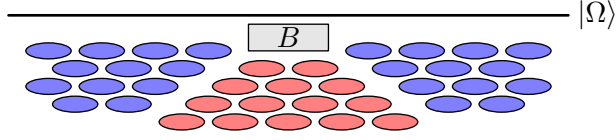


Figure 2: An illustration of the expression $A^\ell B|\Omega\rangle$ in a 1D system. B is a local perturbation, applied to the ground state $|\Omega\rangle$. The local terms underneath it correspond to the P_i projections in A^ℓ . The pink terms are the projections inside the causality cone of B . These terms are graph connected to B , and generally do not commute with it. The blue terms are outside the causality cone, and can therefore commute with B and be absorbed by $|\Omega\rangle$.

The proof hinges on the following observation, which we refer to as the *norm-energy trade off*. Assume by contradiction that A does not move a vector $|\psi\rangle$, which is orthogonal to the ground state, very much. Then $A|\psi\rangle$ must be very close to the range of each P_i , but since the range of P_i is the null space of the local term H_i , this means that the energy $\langle\psi|A^\dagger H_i A|\psi\rangle$ must be small. However, on the other hand, the sum of those energies must be larger than ϵ , since $A|\psi\rangle$ is orthogonal to the ground space; this implies that the shrinkage must be quite significant, providing an upper bound on the norm of the vector $A|\psi\rangle$.

However, the above argument is not sufficiently strong. Since there are m terms H_i , the energy contribution of each term can be as small as ϵ/m ; this will lead to a factor of ϵ/m in the lemma, which is not strong enough. The key point is that the energy-norm trade off can be applied locally, using the tensorial structure of A ; we break the movement of $|\psi\rangle$ to $A|\psi\rangle$ into disjoint sequential steps and then relate the contributions to the energy of *each* of the terms H_i with the shrinkage resulting from each step; one might suspect that entanglement could prevent such an analysis in which shrinkage accumulates but the point is exactly that the local structure of the problem allows this accumulation to happen. Think very simplistically of the state $a|00000\rangle + b|11111\rangle$ subjected to the local terms $H_i = |1\rangle\langle 1|_i$. A projection of this state on the ground state of H_i for any one of qubits i , results in a shrinkage by a factor of $|a|^2$, but once one projection is applied in one location, the shrinkage is exhausted and no more shrinkage is to be gained by a projection in another location. That this entanglement related phenomenon does not happen in the DL scenario is due to the locality of the operators involved; it highlights that the way the state $A|\psi\rangle$ can be entangled is severely limited.

We demonstrate the applicability of this reformulation of the DL by providing significant simplifications of the proof of Hastings' area law in 1D [2], using the DL in two key points, bypassing completely the analytic methods. By this we hope to make this important result accessible to a wider audience, as well as possibly extendable to higher dimensions. The outline of the proof still follows that of Hastings, but now becomes much easier to understand; we defer the explanation of how the proof goes and how the detectability lemma enters the picture to Sec. 5.

To give another example, we provide a one page, very simple proof of Hastings' celebrated result that

the correlations in the ground states of gapped Hamiltonians decay exponentially with the distance [11]. Unlike the area law, this applies to d -dimensional grids for any constant d . More precisely, consider two observables A and B that are local and act on sets of particles that are of distance ℓ on the grid; the decay of correlations means that the expectation value of their product is almost as that of the product of their expectation, up to an error which decays exponentially in ℓ .

We mention that at first sight, one might connect the exponential decay of correlations to an intuition that entanglement between a region L and its surrounding is “located” only close to the boundary of L , and thus scales like the area rather than like the volume. Though an appealing intuition, such an implication of exponential decay of correlation to area laws is not known, and indeed quantum expanders provide a counter-example to such a naïve connection [13].

In both of those proofs, the DL replaces a combination of the Lieb-Robinson bound with other analytic tools; this works of course only when the DL is applicable, namely, for the rich case of frustration-free Hamiltonians. The restriction to frustration-free Hamiltonians may seem quite strong. We note, however, that there are various frustration-free systems that are interesting from a physics and a computational points of view, such as the ferromagnetic XXZ model, the AKLT model [14], and stabilizer codes such as the Toric code [15]. In addition, many of the quantum phenomenon in quantum Hamiltonian complexity are revealed already in the context of frustration-free Hamiltonians, and the major open problems in this area (e.g., quantum PCP and 2D area law) are wide open already for this case. Much is to be learned from studying frustration-free Hamiltonians, before we proceed to the more general case; it seems that the simpler combinatorial nature of the DL in this case might provide a new handle to those questions, and there are reasons to believe that a proof of an area law for frustration-free systems might be extendable to the general case.

To illustrate how exactly the DL is related to the analytic methods, we start our more technical discussion with a toy application comparing the usage of the LR bound to the alternative route offered by the DL, in Sec. 4.

Related work and further directions:

The DL seems to be connected to various diverse scientific areas. The connections to the LR bound and other analytic tools used in condensed matter physics are discussed extensively in Sec. 4; one other connection is to view of the DL operator A as a special instance of the general *Method of Alternating Projections (MAP)*, that was first studied by von Neumann [16]. In that method one applies a fixed sequence of projections in order to approach the intersection subspace. In the general setting, the projections are not assumed to be local, nor the Hilbert space is assumed to be of finite dimension. In recent results [17], the convergence rate is given as a function of the *Fridriechs angle*, which is not easily related to a physical quantity. The DL, on the other hand, is a MAP under the special assumption that the projections are local, associated with a frustration-free k -local Hamiltonian, with a convergence rate that is given as a function of the spectral gap. It would be interesting to see if more insight can be derived from these connections.

Recently, much attention was given to a quantum algorithm which, given a local Hamiltonian, uses a process involving random *measurements* of the energies of the local terms to approach the ground state efficiently (for certain cases) [18, 19]. The algorithm discussed in those papers carries similarities to the situation we are handling here, despite the fact that measurements are applied rather than projections, and also that the terms are chosen randomly, rather than in some fixed order. It seems that the DL lemma, and the energy-norm trade off, could potentially be useful also for the analysis of such algorithms. In particular, it would be very interesting to see a version of the detectability lemma which applies for the case in which the terms are chosen randomly.

As discussed above, it is a wide open question to apply the combinatorial tools presented in this paper to the major open problems of quantum PCP and area laws in dimensions higher than 1, as well as to many other basic open questions in quantum Hamiltonian complexity.

Paper organization:

We start with notations and preliminaries in Sec. 2, and then proceed to the statement and proof of the DL in Sec. 3. In Sec. 4, we provide the example comparing the LR bound approach to the DL one. We then proceed to the area law proof in Sec. 5, and conclude with the one page proof of the exponential decay in Sec. 6.

2 Notations and Preliminaries

We consider a k -local Hamiltonian H acting on $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$, the space of n particles of dimension d . $H = \sum_i H_i$ where each H_i is a non-negative and bounded operator that acts non-trivially on a constant number of k qubits (hence the term local Hamiltonian). We assume that H has a ground space of energy 0, which must therefore also be a common zero eigenspace of all terms H_i . This means that H is frustration free. We also assume that H is “gapped”, meaning that its lowest eigenvalue is 0 (the ground energy) and all the next are equal or larger than some constant $\epsilon > 0$. We denote by $\mathcal{H}' \subset \mathcal{H}$ the orthogonal complement ground space of H . Thus \mathcal{H}' is an invariant subspace for H , and

$$H|_{\mathcal{H}'} \geq \epsilon \mathbb{1} . \quad (2)$$

Most of these assumptions, except for perhaps the frustration-free assumption, are very often used in condensed matter physics.

Throughout this paper we further assume that the H_i ’s are projections, and hence would be denoted by Q_i . We define P_i to be the projection on the ground space of Q_i , $P_i \stackrel{\text{def}}{=} \mathbb{1} - Q_i$. The assumption that H is made of projections is not actually a restriction because we can reduce any frustration-free, bounded and gapped system into that case. Specifically, for $H = \sum_i H_i$ with $\|H_i\| \leq K$, and a spectral gap $\tau > 0$, we first add an appropriate constant to each H_i such that their ground energy is 0. Then for every i we define Q_i as the projection into the space where the energy of H_i is greater than 0 and $P_i \stackrel{\text{def}}{=} \mathbb{1} - Q_i$ as the projection to the ground space of H_i . Finally, we define the auxiliary Hamiltonian $H' = \sum_i Q_i$. This

system is frustration free because the original ground states would also be ground states in H' with a vanishing energy. Moreover, for any state $|\psi_\perp\rangle \in \mathcal{H}'$ and every H_i ,

$$\langle \psi_\perp | H_i | \psi_\perp \rangle = \langle \psi_\perp | Q_i H_i Q_i | \psi_\perp \rangle \leq K \langle \psi_\perp | Q_i | \psi_\perp \rangle ,$$

and therefore the gap in H' is $\epsilon \geq \tau/K$. It follows that all of our results can be applied to bounded frustration-free Hamiltonians by replacing the gap ϵ in DL with the scaled version τ/K .

Given a state $|\phi\rangle$ and a partition of the qubits to two non intersecting sets, R and L , with corresponding Hilbert spaces $\mathcal{H}_L, \mathcal{H}_R$, we can consider the Schmidt decomposition of the state along this cut: $|\phi\rangle = \sum_j \alpha_j |L_j\rangle \otimes |R_j\rangle$. Here $\alpha_1 \geq \alpha_2 \geq \dots$ are the *Schmidt coefficients*. Their squares are equal to the non-zero eigenvalues of the reduced density matrices to either side of the cut $\rho_L(\phi)$ and $\rho_R(\phi)$, which we denote by $\lambda_1 \geq \lambda_2 \geq \dots$. The *Schmidt rank* of $|\phi\rangle$ is then the number of non-zero eigenvalues λ_j (or Schmidt coefficients α_j), and the *entanglement entropy* is the entropy of the set $\{\lambda_i\}$, or, equivalently, the von Neumann entropy of the matrix $\rho_L(\phi)$. A straightforward corollary of the Eckart-Young theorem [20] is then that the truncated Schmidt decomposition provides the best approximation to a vector in the following sense:

Fact 2.1 *Let $|\phi\rangle$ be a vector on $\mathcal{H}_L \otimes \mathcal{H}_R$, and let $\lambda_1 \geq \lambda_2 \geq \dots$ be the eigenvalues of its reduced density matrix. The largest inner product between $|\phi\rangle$ and a norm one vector with Schmidt rank r is $\sqrt{\sum_{j=1}^r \lambda_j}$.*

3 The detectability lemma: A new proof

For clarity of presentation, we will prove the DL in the case stated in the introduction: where the particles are set on a line and the local terms are two-local involving nearest neighbors. This proof contains all the necessary ingredients for the proof of the more general DL in the case where the Hamiltonian has k local terms that can be partitioned into g layers; we make the precise statement of the more general case at the end of this section.

We begin with a simple lemma that quantifies the norm-energy trade-off in the simple case of two projections X, Y : we show that if the application of XY does not move a vector very much then the energy of that vector with respect to $\mathbb{1} - Y$ must be small:

Lemma 3.1 *Given arbitrary projections X, Y and $|v\rangle$ of norm 1, if $\|XYv\|^2 = 1 - \epsilon$ then*

$$\|(\mathbb{1} - Y)XYv\|^2 \leq \epsilon(1 - \epsilon) . \quad (3)$$

The proof is given in the Appendix. Let us now proceed to prove the detectability lemma.

Proof of Lemma 1.1:

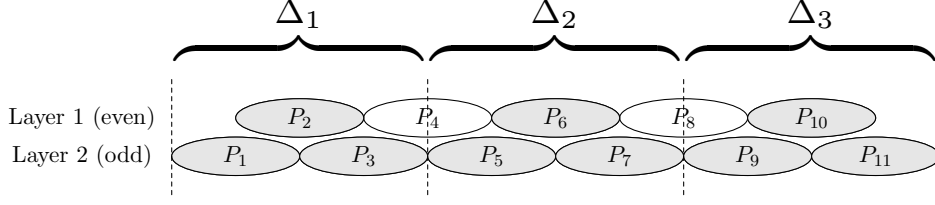


Figure 3:

Suppose $|\psi\rangle \in \mathcal{H}'$ is a norm 1 state that is orthogonal to the ground space, and define $|\phi\rangle \stackrel{\text{def}}{=} A|\psi\rangle$. Notice that for every ground state $|\Omega\rangle$, $\langle\Omega|A|\psi\rangle = 0$ and so $|\phi\rangle$ is orthogonal to the ground space. We would like to show that

$$\|\phi\| \leq \frac{1}{(\epsilon/2 + 1)^{1/3}}. \quad (4)$$

We will find both a lower and upper bounds for the energy of $|\phi\rangle$, $\langle\phi|H|\phi\rangle$, which will give us an inequality for $\|\phi\|^2$, from which Eq. (4) will follow.

The lower bound is straightforward since $|\phi\rangle$ is orthogonal to the ground state, and so

$$\langle\phi|H|\phi\rangle \geq \epsilon\|\phi\|^2. \quad (5)$$

We shall now upper bound the energy $\langle\phi|H|\phi\rangle$ by carefully upper bounding the contributions of the individual terms $\langle\phi|Q_i|\phi\rangle$. We begin by noting that these terms are equal to 0 for i odd since $A = \Pi_{\text{odd}}\Pi_{\text{even}}$ and $Q_i\Pi_{\text{odd}} = 0$ for any odd i (recall that $\Pi_{\text{even}}, \Pi_{\text{odd}}$ are products of the projections $P_i = \mathbb{1} - Q_i$). We now want to bound the contributions coming from the even terms.

For this purpose we present A in a convenient form, by reordering its terms. We call the triplet product of projections $(P_1P_3P_2), (P_5P_7P_6), \dots$ *pyramids*, and denote them by $\Delta_i = P_{4i-3}P_{4i-1}P_{4i-2}$; The remaining terms are combined to the operator $R \stackrel{\text{def}}{=} P_4P_8\dots$. See Fig. 3 for an illustration of this structure in 1D. Notice that by just using the fact that P_i and P_j commute when i and j are not consecutive, we can write:

$$A = \Delta_1\Delta_2\dots\Delta_m R,$$

where m is the number of pyramids which is approximately $n/4$.

We will use this reordering to bound the energy contribution of the terms Q_2, Q_6, \dots ; a symmetric argument will bound the remaining even terms Q_4, Q_8, \dots etc. The energy contribution of Q_{4i-2} will be related to the amount of movement produced by the Δ_i portion of the operator A .

The key point in providing this bound is this. We view the transformation of $|\psi\rangle \rightarrow |A\psi\rangle = |\phi\rangle$ as a series of steps given by the application of the pyramids Δ_i . Specifically, letting $|v_i\rangle \stackrel{\text{def}}{=} \Delta_i\Delta_{i+1}\dots\Delta_m R|\psi\rangle$, we consider the transformation $|\psi\rangle \rightarrow R|\psi\rangle \rightarrow |v_m\rangle \rightarrow |v_{m-1}\rangle \rightarrow \dots \rightarrow |v_1\rangle = A|\psi\rangle$. The square of the

norm of the first state, after applying R , is $a_m \stackrel{\text{def}}{=} \|R\psi\|^2$. Let $a_i \stackrel{\text{def}}{=} \|v_i\|^2 / \|v_{i+1}\|^2$ be the “shrinkage” of the square norm (or movement) resulting from the application of the i th pyramid, for $1 \leq i < m$.

We shall now prove, using Lemma 3.1, that the shrinkage a_i is related to the energy of the operator Q at the top of the same pyramid Δ_i :

$$\langle \phi | Q_{4i-2} | \phi \rangle \leq 1 - a_i .$$

We write

$$\begin{aligned} \langle \phi | Q_{4i-2} | \phi \rangle &= \|(\mathbb{1} - P_{4i-2})A\psi\|^2 = \|\Delta_1 \cdots \Delta_{i-1}(\mathbb{1} - P_{4i-2})\Delta_i v_{i+1}\|^2 \\ &\leq \|(\mathbb{1} - P_{4i-2})\Delta_i v_{i+1}\|^2 , \end{aligned}$$

and recall that $\Delta_i = P_{4i-3}P_{4i-1}P_{4i-2}$. We can therefore apply Lemma 3.1 to $(\mathbb{1} - P_{4i-2})\Delta_i \frac{v_{i-1}}{\|v_{i-1}\|}$ (with $Y = P_{4i-2}$ and $X = P_{4i-3}P_{4i-1}$), and conclude that $\|(\mathbb{1} - P_{4i-2})\Delta_i v_{i-1}\|^2 \leq (1 - \|\Delta_i v_{i-1}\|^2 / \|v_{i-1}\|^2) \|v_{i-1}\|^2 = (1 - a_i) \|v_{i-1}\|^2 \leq (1 - a_i)$. Consequently $\langle \phi | Q_{4i-2} | \phi \rangle \leq 1 - a_i$.

Using this upper bound gives an upper bound for the energy contribution for Q_i , $i \in \{2, 6, 10, \dots\}$:

$$\langle \phi | (Q_2 + Q_6 + \dots) | \phi \rangle \leq \sum_i (1 - a_i) ,$$

with the constraint on the norm $\|\phi\|^2 \leq \prod a_i$. The right hand side is maximized when all the a_i s are equal to each other, i.e., $a_i = \|\phi\|^{\frac{2}{m}}$, and therefore we are left with an upper bound of the energy coming from $Q_2 + Q_6 + \dots$ as:

$$\langle \phi | (Q_2 + Q_6 + \dots) | \phi \rangle \leq m \left[1 - \|\phi\|^{2/m} \right] \leq \frac{1 - \|\phi\|^2}{\|\phi\|} ,$$

where the last inequality follows from the fact¹ that for every $x \in [0, 1]$, we have $m [1 - x^{1/m}] \leq \frac{1-x}{\sqrt{x}}$.

For the energy of $Q_4 + Q_8 + \dots$, a similar decomposition to $A = (P_3 P_5 P_4)(P_7 P_9 P_8) \cdots (P_2 P_6 \cdots)$ can be made, thereby upper bounding the energy by $2 \frac{1 - \|\phi\|^2}{\|\phi\|}$. Combining the energy upper and lower bounds we therefore get

$$\epsilon \|\phi\|^2 \leq \langle \phi | H | \phi \rangle \leq 2 \frac{1 - \|\phi\|^2}{\|\phi\|} , \quad (6)$$

and so

$$\frac{\epsilon}{2} \|\phi\|^3 \leq 1 - \|\phi\|^2 \leq 1 - \|\phi\|^3 , \quad (7)$$

which gives

$$\|\phi\| \leq \frac{1}{(\epsilon/2 + 1)^{1/3}} . \quad (8)$$

■

¹This inequality can be easily verified by noticing that $f_m(x) \stackrel{\text{def}}{=} m\sqrt{x} [1 - x^{1/m}] + x$ is equal to 1 for $x = 1$ and has a non-negative derivative for $x \in [0, 1]$ and $m \geq 1$.

The above proof can be easily generalized to other geometries. In the general case, in accordance with Sec. 2, we assume we have a k -local, frustration-free Hamiltonian $H = \sum_i Q_i$ that is made of projections and has a spectral gap $\epsilon > 0$. We further assume that each particle participates in a constant number of projections, and therefore the Q_i can be partitioned into a constant number of g layers; each layer is made of projections that do not intersect each other and are therefore commuting.

Then for each layer we define the projection Π_i as the product of all $P_j = \mathbb{1} - Q_j$ that are in the layer, and define the DL operator A by

$$A \stackrel{\text{def}}{=} \Pi_g \cdots \Pi_1 . \quad (9)$$

Finally, we define $f(k, g)$ to be the number of *sets* of pyramids that are necessary to estimate the energy contribution of all the Q_i terms. In the 1D case that we proved, we had $f(k, g) = 2$, because only the even layer contributed energy and we needed two sets of pyramids to cover that layer. In the general case it is easy to see that $f(g, k)$ can be crudely bounded by $f(g, k) \leq (g - 1)k^g$.

Using the above definitions, the general DL is

Lemma 3.2 (The detectability lemma) *Consider the local Hamiltonian system that is described above. Then*

$$\|A|_{\mathcal{H}'}\| \leq \frac{1}{[\epsilon/f(k, g) + 1]^{1/3}} . \quad (10)$$

4 Comparing the Lieb-Robinson bound approach and the detectability lemma approach

In this section, we compare the DL with a standard method used in many of the seminal results in quantum Hamiltonian complexity, such as Hastings' areas law for 1D gapped systems [2], and Hastings' exponential decay of correlations proof [11]. The method combines the use of the Lieb-Robinson bound (LR bound), with a Fourier analysis and the existence of a gap, to reveal the locality properties of the ground state. More specifically, the method uses these tools to approximate expressions that involve the projection operator to the ground state, Π_{gs} , by local operators.

To understand how this is done, let us concentrate on a simple example of locality in the ground state, and derive it using both the DL and the LR bound.

We focus on Π_{gs} , the projection on the ground space of H . On the surface, this projector seems very far from being local in any sense. Nevertheless, in gapped systems it does possess some locality properties that are crucial to the analysis of correlations and entanglement in the ground state. To see this, one standardly considers an approximation of Π_{gs} by another operator:

$$P_q \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi q}} \int dt e^{-t^2/2q} e^{-iHt} . \quad (11)$$

Here q is a free parameter to be chosen as appropriate. For an eigenvector $|E\rangle$ of H with eigenvalue E , we have $P_q|E\rangle = e^{-qE^2/2}|E\rangle$. Consequently, if the system has a constant spectral gap $\epsilon > 0$, P_q indeed approximates Π_{gs} well:

$$\|P_q - \Pi_{gs}\| \leq e^{-q\epsilon^2/2} .$$

We now want to argue regarding the local nature of P_q in various contexts. Let us illustrate the LR bound approach with a simple example: consider the expression $\Pi_{gs}B|\Omega\rangle \approx P_qB|\Omega\rangle$, where $|\Omega\rangle$ is a ground state of the system with zero energy, and B is some local perturbation. It is easy to see that

$$P_qB|\Omega\rangle = \frac{1}{\sqrt{2\pi q}} \int dt e^{-t^2/2q} e^{-iHt} B e^{iHt} |\Omega\rangle = \frac{1}{\sqrt{2\pi q}} \int dt e^{-t^2/2q} B(t) |\Omega\rangle . \quad (12)$$

$B(t) \stackrel{\text{def}}{=} e^{-iHt} B e^{iHt}$ is the time evolution of the perturbation B . The key point is now to use the famous LR bound, to approximate it by a “local” operator, i.e., an operator which acts only on the “neighborhood” of the particles on which B acts. The following is an immediate corollary of the original LR bound, which we omit for sake of brevity. The full statement of the LR bound, together with the proof of this corollary can be found in Ref. [21].

Theorem 4.1 (Lieb-Robinson bound (LR bound) , adapted [21]) *Given a local Hamiltonian $H = \sum_i H_i$ on n particles, there exists a constant velocity v s.t. $B(t)$ can be approximated by an operator denoted $B_\ell(t)$ whose support is inside a ball of radius $\ell = vt$ around the support of B , s.t.*

$$\|B(t) - B_\ell(t)\| \leq \|B\| \cdot e^{-\mathcal{O}(\ell)} . \quad (13)$$

Given a length scale $\ell > 0$, we may now set $q = \ell$ in Eq. (12) and obtain

$$\Pi_{gs}B|\Omega\rangle \approx P_\ell B|\Omega\rangle \approx \frac{1}{\sqrt{2\pi\ell}} \int_{|t|\leq\ell} dt e^{-t^2/2\ell} B(t) |\Omega\rangle \approx \frac{1}{\sqrt{2\pi\ell}} \int_{|t|\leq\ell} dt e^{-t^2/2\ell} B_\ell(t) |\Omega\rangle .$$

In the above series of approximations \approx implies an approximation of up to an error of $e^{-\mathcal{O}(\ell)}$. The 1st approximation follows from the assumption of the constant gap and Eq. (11). The 2nd approximation is due to the exponential decay of the filter function $e^{-t^2/2\ell}$, and the 3rd is due to the LR bound. We therefore get an exponentially (in ℓ) good approximation to $\Pi_{gs}B$ in the expression $\Pi_{gs}B|\Omega\rangle$ by an operator which is ℓ -local.

Let us now derive the same result for the frustration-free case using the DL. First, we approximate the ground space projection Π_{gs} by applying the DL operator A for m times. By Eq. (10), A leaves the ground space invariant while shrinking the orthogonal space by a constant factor. Therefore

$$\Pi_{gs} = A^m + e^{-\mathcal{O}(m)} . \quad (14)$$

We now write

$$\Pi_{gs}B|\Omega\rangle = A^m B|\Omega\rangle + \|B\| \cdot e^{-\mathcal{O}(m)} ,$$

and consider the expression $A^m B|\Omega\rangle$. By assumption, the system is frustration free, and therefore every local projection operator P_i that appears in A leaves $|\Omega\rangle$ invariant: $P_i|\Omega\rangle = |\Omega\rangle$. We now consider the “causality cone” of projections in A^m that are defined by B . These are simply the projections that are graph-connected to B when all the projections in A^m are arranged in consecutive gm layers (see Fig. 2). The main observation is that all the projections outside this causality cone commute with B , and can therefore be absorbed by $|\Omega\rangle$. We are therefore left only with the projections of the causality cone, whose support size ℓ is proportional to m . In other words, just as in the LR bound method, we found an exponentially (in ℓ) good approximation to $\Pi_{gs}B$ in the expression $\Pi_{gs}B|\Omega\rangle$ by an operator which is ℓ -local.

This kind of reasoning, with appropriate modifications, is used in both the 1D area-law and the exponential decay of correlations, that are presented in the following sections.

5 The area law in 1D using the detectability lemma

Throughout this section, we let $H = \sum Q_i$ be a 2-local frustration-free 1D Hamiltonian that is made of projections Q_i acting on particles of dimension d . Assume that H has a unique ground state $|\Omega\rangle$ and a spectral gap ϵ , and set $\delta \stackrel{\text{def}}{=} 1 - (1 + \epsilon/2)^{-1/3}$ in accordance with the shrinking exponent of the DL, in Eq. (1). We notice that in the interesting limit $\epsilon \rightarrow 0$, we have $\delta \simeq \epsilon/6$, and generally, for $\epsilon \in [0, 1]$, have $\epsilon/8 \leq \delta \leq \epsilon/6$. In order to keep the presentation simple, we shall assume throughout this section that $\epsilon \leq 1$, and prove the following version of a one dimensional area law:

Theorem 5.1 (Area Law for frustration free Hamiltonians in 1D) *For any contiguous cut along the chain, the entanglement entropy of the ground state $|\Omega\rangle$ across the cut is bounded by a constant which depends on the dimensionality of the particles d and on the spectral gap ϵ ; specifically,*

$$S \leq \frac{10}{\delta} d^{4/\delta} (\ln d)^2 . \quad (15)$$

The proof relies on two main lemmas. The first shows that for any cut along the line, there is a product state $|\phi_1\rangle \otimes |\phi_2\rangle$ that has a constant inner product with the ground state:

Lemma 5.2 (Constant overlap with a product state) *For every cut, there is a product state $|\phi_1\rangle \otimes |\phi_2\rangle$ such that $|\langle \phi_1 \otimes \phi_2 | \Omega \rangle| \geq \mu \stackrel{\text{def}}{=} d^{-\ell} (1 - \delta)^{\ell_0/4}$, with $\ell_0 \stackrel{\text{def}}{=} d^{4/\delta}$.*

The second lemma shows that if there exists a product state with a constant overlap with the ground state $|\Omega\rangle$, then $|\Omega\rangle$ has finite entanglement entropy:

Lemma 5.3 (Constant overlap with a product state implies finite entropy) *If for some cut there exists a product state $|\phi_1\rangle \otimes |\phi_2\rangle$ such that $|\langle \phi_1 \otimes \phi_2 | \Omega \rangle| \geq \mu$, then the entanglement entropy of $|\Omega\rangle$ across the cut is bounded by*

$$S \leq \frac{3}{\delta} \left(\ln \frac{1}{\mu^2 \delta} + 2 \right) \ln d . \quad (16)$$

Theorem 5.1 then follows easily by combining the two lemmas and using the facts that $\delta \geq \mu$ and $\ln \frac{1}{\mu} \geq 6$. We prove Lemma 5.2 in Sec. 5.2 and Lemma 5.3 in Sec. 5.1.

5.1 Constant overlap implies finite entropy (proof of Lemma 5.3)

In this section we prove Lemma 5.3. The DL is clearly the right tool for the task, since it provides a “local” operator that can be repeatedly applied to the promised product state $|\phi_1\rangle \otimes |\phi_2\rangle$ without increasing its entanglement rank much, while exponentially decreasing its distance from the ground state.

The only thing that is not entirely clear is how to get a constant bound on the entanglement entropy of the ground state, since a straightforward argument would mean applying the operator non-constant number of times to get arbitrarily close to the ground state. The key is to observe that after ℓ applications of the DL we get a state with a bounded Schmidt rank that is close to the ground state, and by Fact 2.1, this gives us a bound on the sum of the largest $d^{2\ell}$ Schmidt coefficients *of the ground state*. With these bounds we can find a pessimistic *constant* upper bound on the entanglement entropy. We can now proceed to the more detailed proof.

Consider then a cut in the line between the particles i_0 and $i_0 + 1$, and let Q_{i_0} be the local term in H that involves $i_0, i_0 + 1$. Assume that along that cut, the product state $|\phi_0\rangle \stackrel{\text{def}}{=} |\phi_1\rangle \otimes |\phi_2\rangle$ has a constant projection μ on the ground state $|\Omega\rangle$:

$$|\phi_0\rangle = \mu|\Omega\rangle + |w\rangle, \quad (17)$$

where $|w\rangle \in \mathcal{H}'$, and $\|w\| = (1 - \mu^2)^{1/2} < 1$. We now apply the operator DL operator A ℓ times on $|\phi_0\rangle$. We obtain

$$A^\ell |\phi_0\rangle = \mu|\Omega\rangle + |w^{(\ell)}\rangle,$$

where $|w^{(\ell)}\rangle \in \mathcal{H}'$ and $\|w^{(\ell)}\| \leq (1 - \delta)^\ell$. Let $|v_\ell\rangle$ be the normalized version of $A^\ell |\phi_0\rangle$. Then

$$|\langle v_\ell | \Omega \rangle| \geq \frac{\mu}{\sqrt{\mu^2 + (1 - \delta)^{2\ell}}} = 1 - \mathcal{O}((1 - \delta)^{2\ell}). \quad (18)$$

This means $|v_\ell\rangle$ are exponentially close to the ground state, as a function of ℓ .

How entangled are those states? We notice that at each application of A , the entanglement rank of the state can only increase by a multiplicative factor of d^2 : for every $i \neq i_0$, the projection term P_i in A works entirely left to the cut or entirely right to the cut, thereby not increasing the Schmidt rank of the state. The only projection in A that may increase the rank is P_{i_0} , and as it is a 2-local projection that works on d -dimensional particles, it can at most increase it by a factor of d^2 .² Consequently, the Schmidt rank of $|v_\ell\rangle$ is at most $d^{2\ell}$.

²Given a vector $|v\rangle = \sum_{j=1}^r |L\rangle_j \otimes |R\rangle_j$ of Schmidt rank r and $i < i_0$, $P_i |v\rangle = \sum_{j=1}^r (P_i |L\rangle_j) \otimes |R\rangle_j$ and thus has Schmidt rank bounded by r (the symmetric argument shows the same result for $i > i_0$). For $i = i_0$, we can decompose $P_{i_0} = \sum_{k=1}^{d^2} X_k \otimes Y_k$ with X_k acting on the i_0 'th particle and Y_k acting on the $(i_0 + 1)$ 'th particle. Consequently $P_{i_0} |v\rangle = \sum_{j=1}^r \sum_{k=1}^{d^2} (X_k |L\rangle_j) \otimes (Y_k |R\rangle_j)$ has Schmidt rank no larger than rd^2 .

We have therefore obtained a family of states $\{|v_\ell\rangle\}$ with Schmidt ranks bounded by $d^{2\ell}$, which are closer and closer to $|\Omega\rangle$. Then using Fact 2.1, together with Eq. (18), it follows that the eigenvalues of the reduced density matrix of $|\Omega\rangle$ along the cut, $\lambda_1 \geq \lambda_2 \geq \dots$, must satisfy

$$\frac{\mu}{\sqrt{\mu^2 + (1-\delta)^{2\ell}}} \leq |\langle v_\ell | \Omega \rangle| \leq \left(\sum_{j=1}^{d^{2\ell}} \lambda_j \right)^{1/2}, \quad (19)$$

which implies the following series of inequalities:

$$\text{For every } \ell \geq 1: \quad \sum_{j \geq d^{2\ell} + 1} \lambda_j \leq \frac{1}{\mu^2} (1-\delta)^{2\ell}. \quad (20)$$

From here, the desired upper bound on the entropy can be deduced by choosing the distribution of maximal entropy which still satisfies the inequalities in Eq. (21). The following lemma, whose proof can be found in Appendix B, gives one such bound:

Lemma 5.4 *Consider a probability distribution $\{\lambda_j\}$ whose values are ordered in a non-increasing fashion, $\lambda_1 \geq \lambda_2 \geq \dots$, and let $D \geq 2$ be an integer and $K \geq 1, 0 < \theta < 1$ some constants such that*

$$\text{for every } \ell \geq 1: \quad \sum_{j \geq D^\ell + 1} \lambda_j \leq K\theta^\ell. \quad (21)$$

Then the entropy of $\{\lambda_j\}$ is upper bounded by

$$S \leq 3 \left[\frac{\ln \frac{K}{1-\theta} + 1}{\ln(1/\theta)} + 2 \right] \ln D. \quad (22)$$

Substituting $D = d^2, \theta = (1-\delta)^2$ and $K = \mu^{-2}$ gives

$$S \leq 6 \left[\frac{\ln \frac{1}{\mu^2[1-(1-\delta)^2]} + 1}{2 \ln[1/(1-\delta)]} + 2 \right] \ln d.$$

But $1 - (1-\delta)^2 \geq \delta$ and $\ln[1/(1-\delta)] \geq \delta$, and so

$$S \leq 6 \left[\frac{\ln \frac{1}{\mu^2 \delta} + 1}{2\delta} + 2 \right] \ln d \leq \frac{3}{\delta} (\ln \frac{1}{\mu^2 \delta} + 2) \ln d,$$

where the last inequality follows from the assumption that $\delta \leq \epsilon/6 \leq 1/6$.

5.2 A product state having constant overlap with $|\Omega\rangle$ (proof of Lemma 5.2)

The obvious candidate for a tensor product state with a constant overlap with $|\Omega\rangle$ is the mixed state $\rho_L \otimes \rho_R$, where ρ_L is the reduced density matrix of $|\Omega\rangle$ to the left of the cut, and ρ_R is the reduced density matrix to the right.

Let us assume for contradiction that the overlap between $|\Omega\rangle$ and $\rho_L \otimes \rho_R$, and in fact with any tensor product state along a certain cut, is less than $(1 - \delta)^{\ell/4}$ for some sufficiently large constant ℓ . If the overlap is small, then there is a measurement that distinguishes $|\Omega\rangle$ from $\rho_L \otimes \rho_R$ with probability of at least $1 - (1 - \delta)^{\ell/2}$; this is simply the projection on the ground state, Π_{gs} .

The challenge is to show that there is a local such measurement, i.e., a measurement confined to a local window, which distinguishes these two states almost as well. Using the DL we shall now find such local measurement that distinguishes with a slightly worse probability $1 - 2(1 - \delta)^{\ell/2}$.

Let us denote by ρ_L^ℓ (respectively ρ_R^ℓ) the reduced density matrix of $|\Omega\rangle$ restricted to the ℓ particles to the left (respectively right) of the cut. Also, let $\rho^{2\ell}$ be $\rho = |\Omega\rangle\langle\Omega|$ restricted to the 2ℓ particles, ℓ on each side of the cut. We refer to the state $\rho_L^\ell \otimes \rho_R^\ell$ as the “disentangled” version of the state $\rho^{2\ell}$. The following lemma shows that under the assumption that $|\Omega\rangle$ has low overlap with every product state $|\phi_1\rangle \otimes |\phi_2\rangle$ (along a given cut), there exists a measurement confined to the window of 2ℓ particles around the cut, that with high probability distinguishes $\rho^{2\ell}$ from $\rho_L^\ell \otimes \rho_R^\ell$.

Lemma 5.5 (Existence of a distinguishing measurement) *Assuming that the overlap of the ground state with any product state satisfies $|\langle\phi_1 \otimes \phi_2|\Omega\rangle| \leq (1 - \delta)^{\ell/4}$, there is a measurement that distinguishes $\rho^{2\ell}$ from $\rho_L^\ell \otimes \rho_R^\ell$ with probability $1 - 2(1 - \delta)^{\ell/2}$.*

The DL ensures that by applying the layers one by one, we converge to the projection on the ground state quickly, and it is this projection that is exactly the distinguishing measurement we want to approximate. We can thus apply A only $\ell/2$ times, approximating the projection on the ground space; now, following the intuition explained in the introduction (and in the example of Sec. 4), only the causality cone of the cut should be used in this measurement, and the rest of the operators in those layers are swallowed by the state being measured; this amounts to a measurement which is restricted to the $-\ell, \ell$ interval and still distinguishes well enough. The detailed proof can be found in the appendix.

The fact that such a measurement exists, distinguishing the original state confined to the 2ℓ window from its “disentangled” version, with high probability, must somehow indicate that there is a lot of entanglement along the cut, whose disentanglement caused this distinguishability. This can be made precise using an information-theoretical argument:

Lemma 5.6 (Distinguishing measurement implies large difference in entropies) *If there is a measurement that distinguishes $\rho^{2\ell}$ from $\rho_L^\ell \otimes \rho_R^\ell$ with probability of at least $1 - 2(1 - \delta)^{\ell/2}$, then*

$$S(\rho_L^\ell) + S(\rho_R^\ell) - S(\rho^{2\ell}) \geq \frac{\delta}{2}\ell - 1 .$$

The lemma implies that the entropy in $S(\rho_L^\ell) + S(\rho_R^\ell)$ is significantly larger than $S(\rho^{2\ell})$, implying that disentangling along the cut has introduced a lot of new entropy. The proof is simple, based on relative entropy; essentially, all it uses is the fact that a measurement that distinguishes with high probability two states, implies high relative entropy between the results of the measurements. Once again, details can be found in the appendix.

To finish the proof of Lemma 5.2, we now need to derive a contradiction. Denote by $S(2\ell)$ the value of $S(\rho^{2\ell})$, with 2ℓ being the segment centered around the cut that provides our contradictory assumption (namely, that any tensor product state has less than $(1 - \delta)^{\ell/4}$ inner product with the ground state). Under these conditions, Lemma 5.5 applies, and hence also the conditions of Lemma 5.6 apply to this segment. Applying Lemma 5.6 we conclude that $S(2\ell) \leq 2S(\ell) - \frac{\delta}{2}\ell + 1$. We now want to recursively apply this inequality, for the ℓ long segments on both sides of the cut, and then for the $\ell/2$ -long segments within those segments, and so on. The problem is that the cuts now move to different locations within the 2ℓ long window, and so our assumption no longer applies for these cuts. However, if the inner product state with any tensor product state is small along the original cut, it can be shown to be quite small also along near-by cuts, and so all the above arguments can be applied for those cuts too. This can be formalized in the following claim, whose easy proof can once again be found in the appendix:

Claim 5.7 *If $|\langle \phi_1 \otimes \phi_2 | \Omega \rangle| \leq \mu$ for all product states across cut $(k, k+1)$ then $|\langle \chi_1 \otimes \chi_2 | \Omega \rangle| \leq \mu d^\ell$ for all product states across any cut $(k+j, k+j+1)$ with $-\ell \leq j \leq \ell$.*

We therefore assume by contradiction that the inner product along a given cut is smaller than μ , such that $\mu d^\ell = (1 - \delta)^{\ell/4}$, and so along all the cuts in the 2ℓ window we have that the inner product of the states is at most $(1 - \delta)^{\ell/4}$, and hence our assumptions apply. We can therefore use the same argument recursively. Since $S(1) \leq \ln(d)$, we get (for ℓ a power of 2), $S(\ell) \leq \ell \ln(d) - \frac{\delta}{2}\ell \log_2 \ell + \ell \leq \ell(\ln(d) + 1) - \frac{\delta}{2}\ell \log_2 \ell$. Choosing ℓ such that $\frac{\delta}{2} \log_2 \ell \geq \ln d + 1$ makes $S(\ell) < 0$ thus giving a contradiction. Using the fact that $d \geq 2$, this can be achieved by $\ell = d^{4/\delta}$ since $d^{4/\delta} \geq 2^{\frac{2(\ln d + 1)}{\delta}}$.

6 Exponential decay of correlations

Consider now a Hamiltonian $H = \sum_i Q_i$ which is k -local and set on a d -dimensional grid. Once again, we assume that Q_i are projections, and that H is frustration free with a unique ground state $|\Omega\rangle$ (i.e., $Q_i|\Omega\rangle = 0$), and a spectral gap $\epsilon > 0$. We wish to show

Theorem 6.1 Decay of Correlations in ground states of gapped Hamiltonians on a d -Dim grid:

Consider a setting as described above. Let X, Y be two local observables whose distance on the grid from each other is m . Denote $\bar{X} \stackrel{\text{def}}{=} \langle \Omega | X | \Omega \rangle$, $\bar{Y} \stackrel{\text{def}}{=} \langle \Omega | Y | \Omega \rangle$. Then

$$|\langle \Omega | (X - \bar{X})(Y - \bar{Y}) | \Omega \rangle| = |\langle \Omega | XY | \Omega \rangle - \bar{X}\bar{Y}| \leq \|X\| \cdot \|Y\| \cdot e^{-\mathcal{O}(m)}. \quad (23)$$

Proof: Let us now consider two operators: P_{in}, P_{out} : P_{in} is defined by applying the DL ℓ times to Y and discarding all projections outside the causality cone of Y . ℓ is chosen such that the resulting cone will not overlap with X (see Fig. 4). Therefore $\ell \propto m$, with the proportionality constant that is a geometrical factor. P_{out} is the complement of P_{in} , i.e., it the layers that one get by applying the DL m times, but

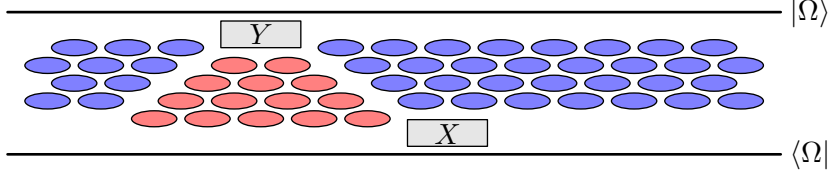


Figure 4: An illustration of the statement $\langle \Omega | XY | \Omega \rangle = \langle \Omega | X P_{out} P_{in} Y | \Omega \rangle$. The operator P_{out} is drawn in blue color and P_{in} is in red. Note that the number of projection layers is proportional to the distance between X and Y .

with a “hole” where the causality cone of Y is. Together, we have $P_{in} \cdot P_{out} = A^\ell$ – See Fig. 4 for an illustration in 1D.

P_{in}, P_{out} leave the ground-state invariant. In addition, they commute with X and Y respectively, hence

$$\langle \Omega | X = \langle \Omega | X P_{in} , \quad Y | \Omega \rangle = P_{out} Y | \Omega \rangle ,$$

and therefore

$$\langle \Omega | XY | \Omega \rangle = \langle \Omega | X P_{in} P_{out} Y | \Omega \rangle = \langle \Omega | X A^\ell Y | \Omega \rangle .$$

We now recall that A^ℓ is in fact an approximation of the ground state projection Π_{gs} (see Eq. (14) in Sec. 4),

$$A^\ell = \Pi_{gs} + e^{-\mathcal{O}(\ell)} = \Pi_{gs} + e^{-\mathcal{O}(m)} ,$$

and so

$$|\langle \Omega | XY | \Omega \rangle - \langle \Omega | X \Pi_{gs} Y | \Omega \rangle| \leq \|X\| \cdot \|Y\| \cdot e^{-\mathcal{O}(m)} .$$

Assuming that the ground state is unique, $\Pi_{gs} = |\Omega\rangle\langle\Omega|$, and therefore

$$|\langle \Omega | XY | \Omega \rangle - \bar{X} \bar{Y}| \leq \|X\| \cdot \|Y\| \cdot e^{-\mathcal{O}(m)} .$$

■

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A Proof of Norm-Energy trade-off, Lemma 3.1

Proof of Lemma 3.1:

Set $|w\rangle \stackrel{\text{def}}{=} Y|v\rangle/\|Yv\|$. Then

$$\|(\mathbb{1} - Y)XYv\|^2 = \|Yv\|^2 \cdot \|(\mathbb{1} - Y)Xw\|^2 .$$

By definition, $|w\rangle$ is a normalized vector inside the support of Y and therefore for every vector $|\psi\rangle$, we have $\|(\mathbb{1} - Y)\psi\| \leq \|(\mathbb{1} - |w\rangle\langle w|)\psi\|$. Plugging this to the equality above, we find

$$\begin{aligned} \|(\mathbb{1} - Y)XYv\|^2 &\leq \|Yv\|^2 \cdot \|(\mathbb{1} - |w\rangle\langle w|)Xw\|^2 = \|Yv\|^2 \cdot \langle w|X(\mathbb{1} - |w\rangle\langle w|)X|w\rangle \\ &= \|Yv\|^2 \cdot \|Xw\|^2 \cdot (1 - \|Xw\|^2) \\ &= \|XYv\|^2 \cdot (1 - \|Xw\|^2) \\ &\leq \|XYv\|^2 \cdot (1 - \|XYv\|^2) = (1 - \epsilon)\epsilon , \end{aligned}$$

where the last inequality follows from the fact that $\|Xw\| \geq \|XYv\|$. ■

B Upper bound on the Entropy

In this section we prove Lemma 5.4, which is required to finish the proof of Lemma 5.3.

Proof:

Call the set of weights $\{\lambda_j\}$ for $D^\ell + 1 \leq j \leq D^{\ell+1}$ the ℓ 'th block. Then the constraints in Eq. (20) imply that for every block $\ell \geq 1$,

$$\sum_{j=D^\ell+1}^{D^{\ell+1}} \lambda_j \leq K\theta^\ell . \tag{24}$$

Obviously, by reshuffling the mass within a block we maintain the constraints. Moreover, it is straight forward to see that the entropy contribution of every block is maximized when all the weights in it are equal. The maximal distribution is therefore a steps function, which satisfies:

$$\text{in block } \ell, \quad \lambda_j \leq \frac{K\theta^\ell}{D^{\ell+1} - D^\ell} = \frac{K}{D-1}(\theta/D)^\ell. \quad (25)$$

We now define ℓ_0 to be the first block for which $K\theta^\ell \leq \frac{1}{e}(1-\theta)\theta$:

$$\frac{\ln \frac{eK}{\theta(1-\theta)}}{\ln(1/\theta)} \leq \ell_0 \leq \frac{\ln \frac{eK}{\theta(1-\theta)}}{\ln(1/\theta)} + 1 = \frac{\ln \frac{K}{1-\theta} + 1}{\ln(1/\theta)} + 2. \quad (26)$$

We will bound the maximal entropy by bounding the entropy contribution of blocks up to (and including) $\ell_0 - 1$ and blocks from ℓ_0 onwards. The first is easy, as there are D^{ℓ_0} weights in the low blocks:

$$S_I \leq \ell_0 \ln D. \quad (27)$$

In the high blocks, $\lambda_j \leq \frac{1}{e}(1-\theta) \leq 1/e$, so we can use the monotonicity of the function $-\lambda \ln \lambda$ in the $(0 : 1/e]$ range to bound the entropy by

$$S_{II} \leq - \sum_{\ell \geq \ell_0} K\theta^\ell \ln \left[\frac{K}{D-1}(\theta/D)^\ell \right] \leq \sum_{\ell \geq \ell_0} K\theta^\ell \ln(D^{\ell+1}) \quad (28)$$

$$= \frac{K\theta^{\ell_0} \ln D}{1-\theta} \left(\ell_0 + \frac{1}{1-\theta} \right) \quad (29)$$

$$\leq \ln D \left(\ell_0 + \frac{1}{1-\theta} \right), \quad (30)$$

where the second equality follows from standard geometric sums identities, and the last inequality follows from the definition of ℓ_0 . Next, looking at the lower bound of ℓ_0 in Eq. (26), it takes standard calculus to verify that $\ell_0 \geq \frac{1+\ln \frac{1}{1-\theta}}{\ln(1/\theta)} + 1 \geq \frac{1}{1-\theta}$, and so $S_{II} \leq 2\ell_0 \ln D$, and

$$S = S_I + S_{II} \leq 3\ell_0 \ln D.$$

Plugging the upper bound of ℓ_0 from Eq. (26), we get Eq. (22). ■

C Proof of Existence of Distinguishing Measurement, Lemma 5.5

Proof of lemma 5.5:

Let $\mathcal{Q} = \{Q_i : Q_i \text{ acts only on particles in the } 2\ell \text{ interval}\}$. Let Π be a projection onto the ground space of all the operators in \mathcal{Q} . We will show that $\{\Pi, 1 - \Pi\}$ is the desired distinguishing measurement.

Clearly $\text{Tr}(\Pi\rho^{2\ell}) = 1$. We would now like to prove that $\text{Tr}(\Pi\rho_L^\ell \otimes \rho_R^\ell)$ is at most $2(1-\delta)^{\ell/2}$.

We start by considering, instead of Π , the applications of the DL operator A .

We can write $\rho_L \otimes \rho_R$ as a convex combination of rank 1 density matrices of product states of the form $|\phi_1\rangle \otimes |\phi_2\rangle$. By assumption, the overlap with the ground state is $|\langle \phi_1 \otimes \phi_2 | \Omega \rangle| \leq (1-\delta)^{\ell/4}$. Therefore, since we assume a unique ground state, $|\phi_1 \otimes \phi_2\rangle = c|\Omega\rangle + (1-c^2)^{1/2}|\Omega_\perp\rangle$, with $c \leq (1-\delta)^{\ell/4}$ and $|\Omega_\perp\rangle$ perpendicular to the ground space. Then by the DL, applying A for $\ell/2$ times, we get $\text{Tr}(A^{\ell/2}|\phi_1\rangle\langle\phi_1| \otimes |\phi_2\rangle\langle\phi_2|) \leq [(1-\delta)^{\ell/4}]^2 + (1-\delta)^{\ell/2} = 2(1-\delta)^{\ell/2}$, and this remains true when we take convex combinations:

$$\text{Tr}(A^{\ell/2}\rho_L \otimes \rho_R) \leq 2(1-\delta)^{\ell/2}.$$

Thus $\text{Tr}(A^{\ell/2}\rho) = 1$, and $\text{Tr}(A^{\ell/2}\rho_L \otimes \rho_R) \leq 2(1-\delta)^{\ell/2}$; this establishes that applying A for $\ell/2$ times distinguishes between ρ and $\rho_L \otimes \rho_R$ with the desired probability. However, we would like the measurement to be confined to a short interval. Since we know that $\text{Tr}(\Pi\rho^{2\ell}) = \text{Tr}(\Pi\rho) = 1$, the proof will follow from showing that

$$\text{Tr}(\Pi\rho_L^\ell \otimes \rho_R^\ell) = \text{Tr}(\Pi\rho_L \otimes \rho_R) = \text{Tr}(\Pi A^{\ell/2}(\rho_L \otimes \rho_R)) \leq \text{Tr}(A^{\ell/2}(\rho_L \otimes \rho_R)).$$

The first equality holds since Π only acts on the 2ℓ particles in $\rho_L^\ell \otimes \rho_R^\ell$ and the last inequality is trivial; it is the middle equality which uses the structure of $A^{\ell/2}$. Indeed, let us write $A^{\ell/2} = A_M A_L A_R$ where $A_M = \underbrace{\cdots (P_{k-2} P_k P_{k+2}) (P_{k-1} P_{k+1}) (P_k)}_{\ell \text{ groups}}$ is a ‘‘pyramid’’ of terms centered at the cut k , and A_L and

A_R are the terms to the left and right of the pyramid respectively, as in Fig. 5. Then $\Pi A^{\ell/2}(\rho_L \otimes \rho_R) = \Pi A_M A_L A_R(\rho_L \otimes \rho_R)$. But since we applied A for exactly $\ell/2$ times, every P_i projection in A_M is also in the 2ℓ window of Π . Therefore $A_M P_i = A_M$ and consequently $\Pi A_M = \Pi$. Similarly, $A_L A_R(\rho_L \otimes \rho_R) = \rho_L \otimes \rho_R$, and therefore $\Pi A^{\ell/2}(\rho_L \otimes \rho_R) = \Pi(\rho_L \otimes \rho_R)$, implying the desired equality. ■

D Proof of Information Theoretical bound, Lemma 5.6

Proof of Lemma 5.6:

Let X and Y be $\{0, 1\}$ random variables that result from applying the measurement Π on ρ and $\rho_L \otimes \rho_R$ respectively. Then by the Lindblad-Uhlmann theorem [22, 23],

$$S(\rho_L^\ell) + S(\rho_R^\ell) - S(\rho^{2\ell}) = S(\rho^{2\ell} || \rho_L^\ell \otimes \rho_R^\ell) \geq S(X || Y) = \sum_{i \in \{0,1\}} x_i \ln \frac{x_i}{y_i}.$$

In this case, we have $X = 1$ with probability 1 and Y is 1 with probability $\alpha \leq 2(1-\delta)^{\ell/2}$. Thus using straight forward analysis, $\sum_i x_i \ln \frac{x_i}{y_i} = \ln(\frac{1}{\alpha}) \geq \ln(1/2) - \frac{\ell}{2} \ln(1-\delta) \geq \frac{\delta}{2}\ell - 1$, and the result follows. ■

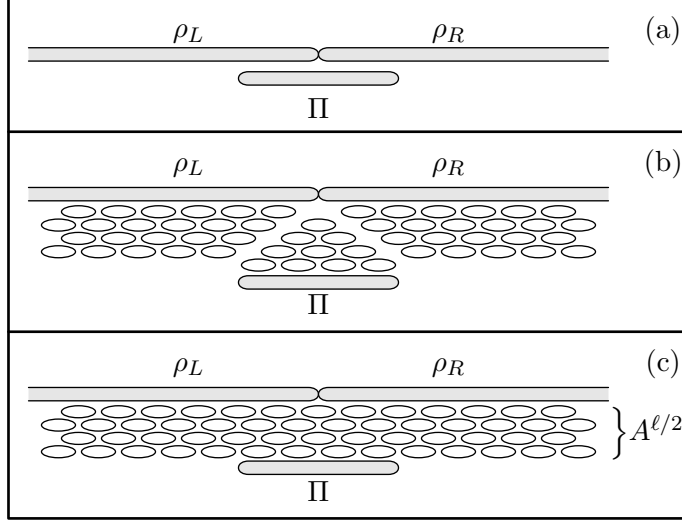


Figure 5: An illustration of the identity $\Pi A^{\ell/2}(\rho_L \otimes \rho_R) = \Pi(\rho_L \otimes \rho_R)$. The left, right and middle sets of projections in Fig. (b) are inside the invariant space of ρ_L, ρ_R and Π respectively.

E Proof that close cuts behave similarly: Claim 5.7

Proof of Claim 5.7:

Assume for contradiction that $|\chi_1\rangle \otimes |\chi_2\rangle$ is a product state across the cut $(k+j, k+j+1)$ with $j > 0$ such that $|\langle \chi_1 \otimes \chi_2 | \Omega \rangle| > \mu d^\ell$. Schmidt-decompose $|\chi_1\rangle = \sum_{i=1}^{d^\ell} \alpha_i |L_i\rangle \otimes |R_i\rangle$ where the cut is between the first k particles and the j particles between $k+1$ and $k+j$. By simple algebra, there exists at least one i such that $|\langle L_i \otimes R_i \otimes \chi_2 | \Omega \rangle| > \mu$ which violates the hypothesis. ■